

Irreversibility in an ideal fluid

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When a real fluid is expelled quickly from a tube, it forms a jet separated from the surrounding fluid by a thin, turbulent layer. On the other hand, when the same fluid is sucked into the tube, it comes in from all directions, forming a sink-like flow. We show that, even for the ideal flow described by the time-reversible Euler equation, an experimenter who only controls the pressure in a pump attached to the tube would see jets form in one direction exclusively. The asymmetry between outflow and inflow therefore does not depend on viscous dissipation, but rather on the experimenter's limited control of initial and boundary conditions. This illustrates, in a rather different context from the usual one of thermal physics, how irreversibility may arise in systems whose microscopic dynamics are fully reversible.

Keywords: jets, irreversibility, Euler equation, boundary conditions, Machian propulsion

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I. INTRODUCTION

Irreversibility is a prominent feature of real-world phenomena. If the video of any but the very simplest process is played backwards, it will soon seem so contrary to experience as to appear comical. For instance, a glass bottle may fall to the floor and shatter into jagged pieces, but we never see the jagged pieces jump up and reassemble themselves into a glass bottle. This irreversibility is captured by the second law of thermodynamics, which states that the entropy (defined to be proportional to the

logarithm of the number of microscopic physical configurations that are macroscopically indistinguishable) tends towards a maximum. This is often summarized, in non-technical terms, as the statement that the randomness (or disorder) of the world increases with time.

The microscopic laws of physics are believed to be time-reversal symmetric, so that nothing actually *forbids* the pieces of the broken bottle from reassembling themselves. The reason why we never see that happen is that it would require an exquisite tuning of the initial conditions of the system: the thermal motions of the molecules in the floor would have to conspire to push the various pieces with just the right velocities to make them converge into the form of the original bottle. If the air, moisture, etc., moved out from the regions of contact between the pieces at the right time, the glass would fuse and a pristine bottle might then land on the table above.

The point is that it is very easy for an experimenter to set up the conditions for a bottle to break, but extraordinarily difficult to set up the conditions for the pieces to reassemble themselves: there are many ways for a bottle to be broken, but comparatively very few for it to be whole.¹ The irreversibility is therefore a consequence of the experimenter's limited control over initial conditions. For a fascinating and accessible discussion of this interpretation of irreversibility and of the problem of the "arrow of time" in statistical mechanics, see [1].

This problem of the arrow of time is usually posed in the context of *thermal* systems, in which mechanical energy can be dissipated into the random motion of the

¹ This explanation, of course, necessarily invokes the experimenter's *subjective* perceptions, because it is the experimenter who describes the physical system in terms of bottles, whole or broken. Technically speaking, the condition of the bottle is a *macrostate*, defined by an experimenter who is insensitive to a great many of the details of the objective, microphysical configuration of the system.

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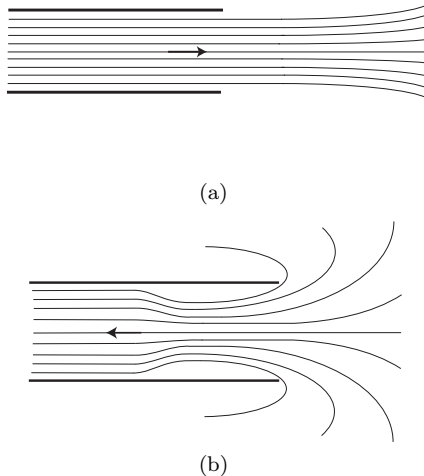


FIG. 1: Streamlines for a real fluid (a) expelled from the mouth of a tube, and (b) aspirated into the mouth of the tube.

particles in a “heat bath.” In this article we will describe an example of a non-dissipative system in which irreversibility nonetheless appears for the same qualitative reason as in statistical mechanics: because of the limited control of the experimentalist over the initial and boundary conditions. Our example will be the flow of an ideal fluid into and out of a tube.

II. MACHIAN PROPULSION

Ernst Mach appears to have been the first experimenter to report that if a solid device alternately aspirates and expels fluid through a single opening, it moves in the same direction as if it only expelled the fluid [2]. In other words, the aspiration does not cancel the momentum gain from the expulsion. In France, this observation has been called the “paradox of Bergeron,” after mechanical engineer Paul Bergeron [7]. It became notorious because of its role in an often-told story from the early life of the eminent theoretical physicist Richard Feynman [3–6]. This phenomenon has recently been called “Machian propulsion” and explained in terms of momentum conservation. [8]

In his delightful *Physics for Entertainment*, originally published in Russian in 1913, Yakov Perelman expresses surprise that this method of propulsion works at all and that it is actually used by squids, jellyfish, and other aquatic creatures, even though it might seem, at first, as absurd as Baron Munchausen lifting himself out of the swamp by pulling on his own hair [9]. The key point is that the outflow can efficiently transfer momentum away from the solid device and into the surrounding fluid medium, while the inflow can do so only by viscous diffusion, because the incoming flow impinges on the inside

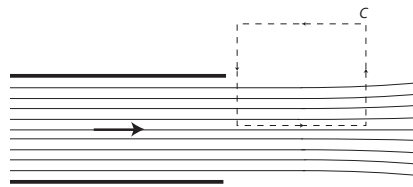


FIG. 2: The integral of $d\mathbf{s} \cdot \mathbf{v}$, along the path C , does not vanish for a jet flow like the one pictured in Fig. 1(a).

of the solid cavity. In fact, it is reverse propulsion during steady aspiration of an ideal fluid that would be akin to the Baron pulling himself up by the hair (see [8]). In mechanical engineering, engines driven by Machian propulsion have been classified as “valveless pulse jet” devices. [10]

Mach noted that there is an asymmetry between the outflow and the inflow, since the outflow forms a narrow jet, whereas the inflow comes in from all directions, as shown in Fig. 1. Many authors have claimed that the omnidirectionality of the inflow is the reason why its momentum does not cancel the momentum of the outgoing jet: see, e.g., [11–13]. This line of argument, however, can be misleading, as discussed in detail in [8].

Even though the asymmetry of the flows in Fig. 1 does not *explain* Machian propulsion, the asymmetry is real and the reason for it deserves a better elucidation than can be found in the literature. It is well-known that “a match can be extinguished by blowing, but not by sucking” [14], because the air only forms a narrow jet when blown out, and that this is related to the fluid’s viscosity. But a more careful consideration of this question reveals conceptually interesting points about the (ir)reversibility of the flows.

III. REAL JETS

Kelvin’s circulation theorem establishes that ideal flows (i.e., those without viscosity) that are irrotational at some initial time remain irrotational for all future times. That is, if the flow is characterized by a velocity field $\mathbf{v}(t, x, y, z)$, then

$$\nabla \times \mathbf{v} = 0 \quad (1)$$

for all t . By Stokes’s theorem, this implies that

$$\oint_C d\mathbf{s} \cdot \mathbf{v} = 0 \quad (2)$$

for any closed path C , defined within a region of space filled with a homogenous ideal fluid.² It is clear from

² Volume II of the *Feynman Lectures on Physics* contains an excellent introduction to ideal flow (what Feynman calls “the flow of dry water”). [15]

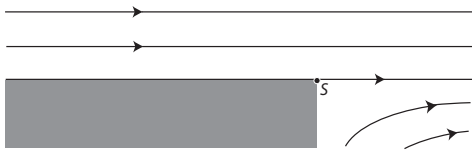


FIG. 3: Closeup of the outgoing flow in Fig. 1(a), in the region around the lip of the tube's mouth. The viscous boundary layer separates from the tube wall at S , where the solid has a sharp edge. This illustration is adapted from [19].

Fig. 2 that Eq. (2) fails for a jet surrounded by fluid that is approximately at rest, so that jet formation must depend on viscosity.

The role of viscosity in the formation of jets may be confusing to a beginner. Tritton writes that “at low Reynolds numbers [i.e., when viscosity is *dominant*], the fluid from an orifice spreads out in all directions. At high Reynolds numbers [i.e., when viscosity is mostly *negligible*] a jet, like a wake, is long and thin” [16]. But even at high Reynolds numbers, the viscosity cannot be ignored within a “boundary layer,” close the solid surface of the tube. This boundary layer plays a key role in the formation of jets in real fluids.

In the boundary layer, the fluid's velocity drops to zero as one approaches the solid surface.³ When the fluid moves along an adverse pressure gradient (i.e., from lower to higher pressure), the streamlines in the boundary layer may break away from the solid, enclosing a region of slow and irregular flow, around which the flow changes direction. This phenomenon is known as *boundary layer separation*, a subject treated in detail in [18].

The flow that leaves the tube's mouth in Fig. 1(a) slows down, and therefore the pressure gradient must be adverse. Figure 3 illustrates the separation of the boundary layer that occurs at a point S , on the lip of the tube's mouth. The fluid caught in the area around S , where the flow changes direction, forms a thin, turbulent layer that separates the jet from the surrounding fluid.

On the other hand, the flow that enters the tube in Fig. 1(b) moves from high to low pressure (i.e., along a favorable pressure gradient) and the boundary layer therefore remains attached to the tube. This explains why, for a real fluid, the outflow forms a jet, whereas the inflow does not.

³ Feynman offers a lovely illustration of the reality of this phenomenon in [17]: Over time, the blades of a powerful industrial fan accumulate a layer of fine dust, which cannot be blown off.

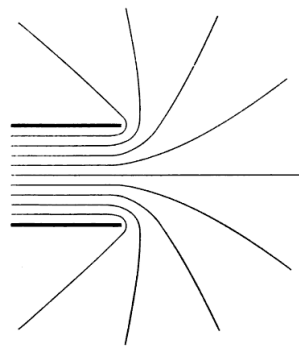


FIG. 4: Streamlines for the irrotational, incompressible flow into or out of tube, within which the flow becomes parallel. The spacing of adjacent streamlines corresponds to an equal flux. The image is by Sir Horace Lamb. [21]

IV. SINKS AND SOURCES

If the flow is both irrotational (Eq. (1)) and incompressible ($\nabla \cdot \mathbf{v} = 0$), then

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla (\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v} = -\nabla^2 \mathbf{v} = 0. \quad (3)$$

The fluid velocity therefore obeys Laplace's equation ($\nabla^2 \mathbf{v} = 0$), whose solutions have been thoroughly investigated in a variety of contexts. If the steady flow within the tube is parallel, the only possible configuration consistent with Laplace's equation is that shown in Fig. 4, regardless of whether the flow is entering or leaving the tube.⁴

Far from the tube, this approaches the shape for a pure sink or a source of flow, with the velocity of the flow becoming radial and falling off, in three dimensions, as $1/r^2$ (see [22]). As we have seen in Sec. III, real fluids can sustain sink-like but not source-like flows, because the latter are destroyed by the separation of the boundary layer at the lip of the tube's mouth. But even in the ideal limit, a source-like flow would be difficult to set up and maintain, because it would require that the experimenter control the fluid pressure not just within the tube, but also at the boundary of the tank that encloses the entire fluid, as we will explain in Sec. VI.

V. TIME REVERSAL FOR IDEAL FLOW

The Euler equation for an ideal fluid of density ρ moving under the action of a pressure field P ,

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\frac{\nabla P}{\rho}, \quad (4)$$

⁴ For instance, Pippard uses this to characterize the form of the radiation from the open end of an acoustic resonator, by analogy to an ideal flow. [20]

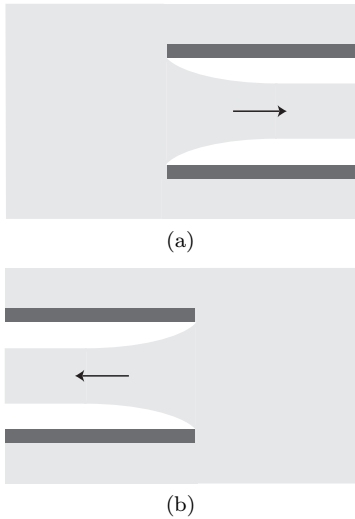


FIG. 5: An ideal fluid, represented in light grey, flows out through a submerged (or “reentrant”) tube. The flow in the tube forms a *vena contracta* (“contracted vein”), with half the tube’s cross-section. The pressure gradient ∇P is trivially reversed between (a) and (b) by reversing the spatial coordinates. In both cases the pressure is lower inside the tube than in the bulk.

is invariant under the time-reversal transformation

$$\mathbf{v}(t, x, y, z) \rightarrow -\mathbf{v}(-t, x, y, z). \quad (5)$$

Thus, for a given pressure gradient ∇P , if a steady flow \mathbf{v} is a solution to Euler’s equation then the reverse flow $-\mathbf{v}$ will be a solution as well. An immediate consequence of this symmetry is that, for an ideal fluid, both the sink-like and the source-like flows imply the same pressure gradient, with higher pressure outside the tube than inside it.⁵

Note that reversing the pressure gradient

$$\nabla P \rightarrow -\nabla P \quad (6)$$

does *not* reverse the flow, which might at first seem counterintuitive. The velocity field \mathbf{v} is defined as a function of time t and absolute position (x, y, z) , but the fluid elements (to which Newton’s laws apply directly) do *not* have fixed positions, since they move along the streamlines. As is very clearly explained in [15], this is the reason for the second term in the left-hand side of Eq. (4). (This is also the source of most of what appears confusing to a student who confronts fluid mechanics for the first

⁵ This also implies that the Feynman reverse sprinkler of [3–6] is *not* the time-reversal of the regular sprinkler that expels water in jets, but rather of a hypothetical submerged sprinkler in which the outflow spreads out and slows down under an adverse pressure gradient (and therefore does not carry any momentum away to infinity).

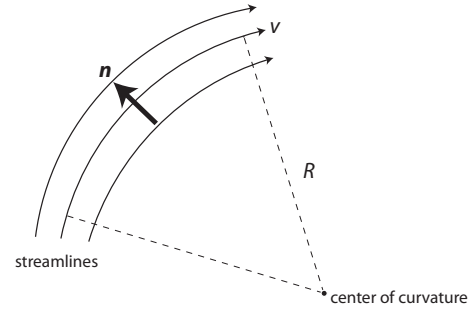


FIG. 6: When the streamlines in a flow turn with radius of curvature R , the pressure gradient along a unit vector \mathbf{n} normal to the streamlines is given locally by Eq. (7). Pressure increases as one moves *away* from center of curvature.

time.) Reversing the force exerted by the pressure gradient on the individual fluid elements changes the streamlines, rather than merely reversing their direction.

Consider, for example, the ideal flow into a submerged tube, as shown in Fig. 5(a). This problem, the “Borda mouthpiece,” is discussed in detail in [8]. The fluid is accelerated by the difference between the higher pressure in the bulk and the lower pressure within the tube. We could trivially reverse ∇P by a coordinate transformation $(x, y, z) \rightarrow (-x, -y, -z)$, but this would also reverse the position of the tube: we would merely have the fluid going out the other way, as in Fig. 5(b). If we made the pressure of the fluid *inside the tube* higher than the pressure in the bulk, the flow pattern would have to change. This is obvious from the fact that (by Bernoulli’s theorem and the condition of incompressibility) the flow must narrow as the pressure drops. We will describe the resulting “ideal jet” in Sec. VIII.

VI. TRANSVERSE PRESSURE GRADIENTS

Whenever the streamlines of a flow curve, there must be a pressure gradient along the direction normal to the streamlines, as illustrated in Fig. 6. It is the force exerted by this pressure gradient that steers the flow along the curving streamlines, overcoming the fluid’s inertia.⁶ If \mathbf{n} is a unit vector normal to the streamlines of the flow at a given point, we have that

$$\mathbf{n} \cdot \nabla P = \frac{\rho v^2}{R}, \quad (7)$$

where R is the radius of curvature of the streamlines. For the flow in Fig. 4, in which R is very small near the lip

⁶ This, rather than Bernoulli’s theorem (which applies to points along the *same* streamline) is what really accounts for the lift on an airplane wing. See [23].

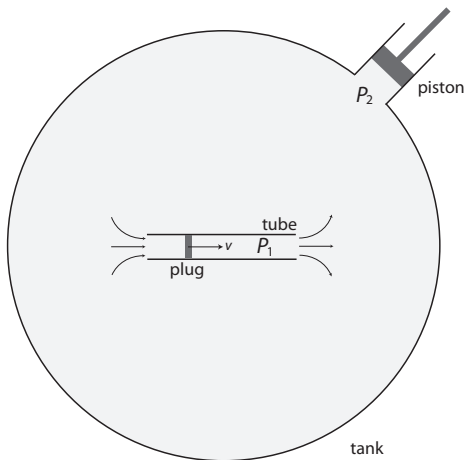


FIG. 7: A long and narrow tube, placed inside a large tank filled with fluid, is open at both ends and contains a movable plug. The pressure at the tank's boundary can be controlled by a piston. This diagram is adapted from an illustration by Sir Horace Lamb in [24].

of the tube's mouth, this means that the pressure of the fluid that is nearly at rest along the boundary of the tank that contains it, must greatly exceed the pressure of the fluid along the edge of the tube's mouth.

Consider the problem of a long and narrow tube, open at both ends, placed inside of a tank filled with an ideal fluid. Let the tube contain a movable plug. If this plug is moved along the tube with a constant speed v , then the only possible steady, irrotational flow that can result is that shown in Fig. 7, with a sink-like flow into one end of a tube and a source-like flow out of the other end. The pressure of the parallel flow within the tube is labelled P_1 . As the flow leaves the tube's mouth and spreads, its speed decreases and its pressure increases. By Bernoulli's theorem, the fluid at rest near the boundary of the tank will have a pressure

$$P_2 = P_1 + \frac{1}{2}\rho v^2. \quad (8)$$

Lamb analyzed this configuration in some detail in [24], from where we have taken the diagram in Fig. 7. As he notes, maintaining this pattern of flow as the plug is moved down the tube would require that the experimenter maintain the high pressure P_2 at the boundary of the tank, which could be done by pushing on the piston connected to the tank, as shown in Fig. 7. But if the experimenter let the piston recede, the streamlines of the fluid that leaves the tube would not be able to spread outwards, because there would not be a sufficient pressure gradient to steer the streamlines around the tube's mouth. Lamb concludes that, in that case, an annular cavity would form at both ends of the tube [24]. A cavity (or bubble) is a region that contains no fluid and therefore effectively exerts a negative pressure (see [25]).

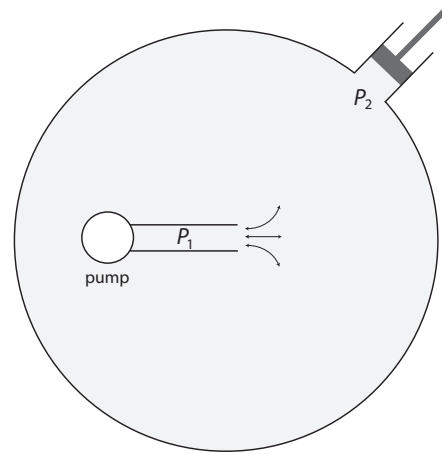


FIG. 8: Instead of the configuration in Fig. 7, we now consider attaching one end of the tube to a pump. It is easy to establish a sink-like flow by lowering the pump's pressure, but maintaining a source-like flow would require control of the pressure P_2 at the tank's boundary.

VII. PUMP-DRIVEN FLOWS

So far, nothing in our analysis has challenged the symmetry between ideal inflows and outflows implied by the Euler equation. In the process represented by Fig. 7, a given fluid element moves from a region of higher pressure P_2 to a region of lower pressure P_1 and finally back to P_2 . The direction of this motion can be reversed by reversing the motion of the plug. But Lamb's analysis reveals that the shape of the flow around the tube's openings depends on the pressure at the tank's boundary.

Suppose that, instead of moving a plug, the experimenter controls a pump attached to one end of the tube inside the tank, as shown in Fig. 8. Fluid can be expelled by increasing the pressure in the pump relative to the fluid at rest in the tank. Conversely, fluid may be sucked in by lowering the pump pressure. This is the case in all of the devices that exhibit Machian propulsion.

It is easy to establish a sink-like flow with such a pump. Given a tank pressure P_2 and a flow speed v , the pump can provide the necessary pressure difference in order to keep the flow within the tube at a lower pressure P_1 that obeys Eq. (8). But producing a source-like flow would be considerably more difficult. With an arrangement such as the one shown in Fig. 8, fluid can only be pushed down the tube by first rising the pump pressure relative to the tube's P_1 . Once the flow in the tube has been established, the experimenter would have to act on the piston in order to obtain a higher pressure P_2 near the tank's inner boundary, in accordance with Eq. (8). Otherwise, the streamlines leaving the tube's mouth would not curve outwards and a cavity would form around the tube's mouth. In Sec. VIII we will describe what would happen to the ideal flow in that case.

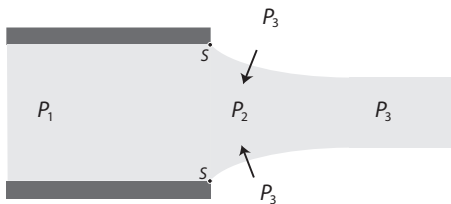


FIG. 9: An ideal, incompressible fluid at high pressure P_1 exits through a tube and into a region where the fluid is at rest at a lower pressure P_3 . Only the moving fluid is shaded. Upon leaving the tube, the flow has an intermediate pressure P_2 . Since $P_2 > P_3$, there is a transverse pressure gradient in the direction of the arrows, and the streamlines therefore curve inwards in that region. The sharp turning of the streamlines at the points marked S requires the formation of a cavity around the tube's mouth.

A mathematician thinks of Eq. (4), together with the condition of incompressibility, as determining both \mathbf{v} and ∇P , for a given choice of initial and boundary conditions on the velocity field \mathbf{v} (see [26, 27]). But ordinarily an experimenter can establish a flow only by manipulating certain pressure differences. Evidently, an experimenter who controls only the pump in Fig. 8 will not be able to reverse the flow in the tank. Similarly, in the case of Machian propulsion, as summarized in Sec. II, the inflow and outflow phases, in which the sign of the pressure difference is flipped, are not time-reversed images of each other even in the limit of zero dissipation.

VIII. IDEAL JETS

If the pressure P_2 of an ideal flow as it leaves the tube is higher than the ambient pressure P_3 of the surrounding fluid at rest, then the only possible steady flow is that pictured in Fig. 9. The transverse pressure gradient from P_2 to P_3 makes the streamlines converge as they leave the tube, until the pressure of the moving fluid drops to P_3 as the flow becomes parallel.⁷

Note that the streamline turns sharply inward as it passes near the point S in Fig. 9, at the lip of the tube's mouth. This requires a very large transverse pressure gradient there. The only way in which the flow itself can create that pressure gradient is to form a cavity near S , where the pressure becomes negative [24]. Thus, in the limit in which the viscosity of the fluid in Fig. 3 vanishes, the separation of the boundary layer at S is replaced by the appearance of a cavity. The discontinuity introduced

by that cavity is what allows us to evade Kelvin's circulation theorem (see Sec. III), which forbade jets from forming in ideal flows.

IX. CONCLUSIONS

The Navier-Stokes equation for real fluids,

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla - \nu \nabla^2\right) \mathbf{v} = -\frac{\nabla P}{\rho} \quad (9)$$

has no time-reversal symmetry. This is widely understood to result from the fact that a kinematic viscosity $\nu > 0$ indicates that the macroscopic energy of the flow may be irreversibly dissipated into the random thermal motion of the fluid's microscopic components. But even an ideal flow with $\nu = 0$ can exhibit a certain kind of irreversibility.

An experimenter who controls only the pressure of a pump connected to a solid tube will see jets form when the fluid is expelled from the tube, never when the fluid is aspirated. Conversely, the experimenter will be able to set up and maintain a sink-like flow into the tube, but not a source-like flow out of it: the pattern in Fig. 4 is plausible only if the streamlines point into the tube. The reason for this is that the source-like flow would require control of the pressure of the fluid not just at the pump attached to the tube, be also at the distant boundary of the tank that receives the fluid, a boundary where the pressure must be kept higher than inside the tube.

Alternatively, we may note that a jet —either a real one as in Fig. 1(a) or an ideal one as in Fig. 9— carries momentum away to infinity (or to the boundary of the surrounding tank). Reversing this motion therefore requires that momentum flow *in* from infinity (or from the tank walls). Setting this up would obviously demand a control of the fluid that an experimenter could not achieve with only a pump attached to the tube.

This irreversibility is similar in spirit to that of statistical mechanics, which also results from the experimenter's limited control of initial and boundary conditions. What is remarkable is that it can be seen even in an ideal flow, without invoking any concept of heat, temperature, or entropy. A video of an ideal fluid moving past the mouth of a tube would, if played backwards, appear very suspect, even though the second law of thermodynamics would play no part in the flow.

The Machian propulsion described in Sec. II can be entirely understood in terms of momentum conservation and is *most* pronounced in the limit of zero viscosity. Even without dissipation, the outflow and inflow phases are not time-reversed images of each other: it is only the pressure difference between the inside and the outside of the solid device that changes sign. Machian propulsion is therefore an example of nonthermal irreversibility.

⁷ It does not violate Bernoulli's theorem for the same pressure P_3 to apply to the ambient fluid at rest and for the parallel flow away from the tube. The reason is that Bernoulli's theorem applies only to points along the *same* streamline, as is very lucidly explained in [23].

Acknowledgments

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